Caringbah High School

YEAR 12 MATHEMATICS
SEMESTER 1 EXAMINATION
2009

Time Allowed: 2 hours + 5 minutes reading time

Instructions

- **All questions must be attempted.**
- **Start each question on a new page.**
- **Answers without necessary working and diagrams may not attract full marks.**
- **Marks may not be awarded for careless or badly arranged work.**
- **Diagrams and working which lead to a correct solution may attract some marks.**
- **Approved calculators may be used**

Question 1 (7 Marks) **Marks**

a) Find the primitive function when

i) $\dfrac{dy}{dx} = 2x^3 - 3$ **1**

ii) $f'(x) = \dfrac{4}{x^2}$ **1**

iii) $\dfrac{dy}{dx} = \sqrt[3]{x}$ **1**

b) Find the sum of the series $6 + 11 + 16 + \ldots + 156$. **2**

c) The sum of the first two terms in an arithmetic sequence is 20. The common difference between successive terms is 4. Find the sum of the first three terms. **2**

Question 2 (7 Marks)

a) Find the second derivative of $y = 4x^3 - 2\sqrt{x}$. **2**

b) Find the gradient of the curve $y = 5x - x^2$ at the point where $x = 2$. **2**

c) State the coordinates of the focus of the parabola $x^2 = 2y$. **1**

d) Evaluate $\int_1^3 (3x^4 + 1)\,dx$ **2**

Question 3 (7 Marks)

a) Find $\displaystyle\int \frac{5x^2 + 1}{x^2}\,dx$ **2**

b) i) Sketch, without using calculus, the graph of $y = \sqrt{2x + 1}$, showing intercepts on the x and y axes. **2**

ii) Calculate the area bounded by the curve $y = \sqrt{2x + 1}$ and the coordinate axes. **3**

Question 4 (7 Marks) **Marks**

a) $\int (2x+5)^3 \, dx$ **1**

b) For the curve $y = x^3 - 3x^2 + 2$,

i) Show that stationary points exist on the curve at $x = 0$ and $x = 2$. **2**

ii) State the nature of each of these stationary points. **2**

iii) Show that a point of inflection occurs at (1,0). **1**

iv) Sketch the graph of $y = x^3 - 3x^2 + 2$. **1**

Question 5 (7 Marks)

a) A company invests \$60 000 in equipment. The value of the equipment depreciates by 15% p.a. What is the equipment worth after 10 years? **1**

b) A man planted a tree that was 1.2m tall and it grew 90cm in the first year. Each subsequent year it grew 20% less than it did in the previous year.

i) How much did it grow in the 5th year after it was planted? **2**

ii) Calculate the height of the tree 5 years after it was planted. **2**

iii) Show that the maximum height that the tree can grow is 5.7m **2**

Question 6 (7 Marks)

a) For the parabola $x^2 = 8y - 8$, find

i) The coordinates of the vertex. **1**

ii) The coordinates of the focus. **1**

iii) The equation of the directrix. **1**

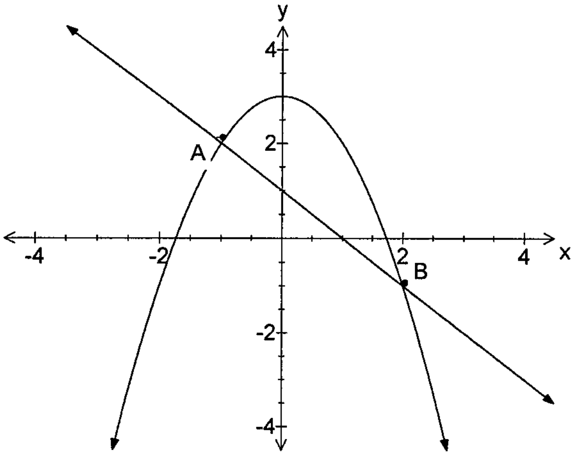
b) Find the equation of the locus of a point that moves such that its distance from the point (2,3) is equal to its perpendicular distance from the line y = -1. **2**

c) Find the volume of the solid formed when the area between the curve $y = x^3$, the y-axis and the lines $y = -8$ and $x = 8$ is rotated about the y-axis. **2**

Question 7 (7 Marks) **Marks**

a) Find the equation of the line that is equidistant from the points A(-1,2) and B(5,4). **2**

b) This diagram shows the line $x + y - 1 = 0$ intersecting with the parabola $y = 3 - x^2$ at the points A and B.



i) By solving the equations simultaneously, show that the coordinates of A and B are (-1,2) and (2,-1) respectively **2**

ii) Find the area enclosed by the parabola $y = 3 - x^2$ and the line $x + y - 1 = 0$. **3**

Question 8 (7 Marks)

a) For the function $f(x) = x^3 - 12x$, state the domain where $f(x)$ is increasing. **2**

b) Two positive numbers have a sum of 24. Find the two numbers if the sum of their squares is to be a minimum. **3**

c) The following table lists the values of a function for 5 values of x. **2**

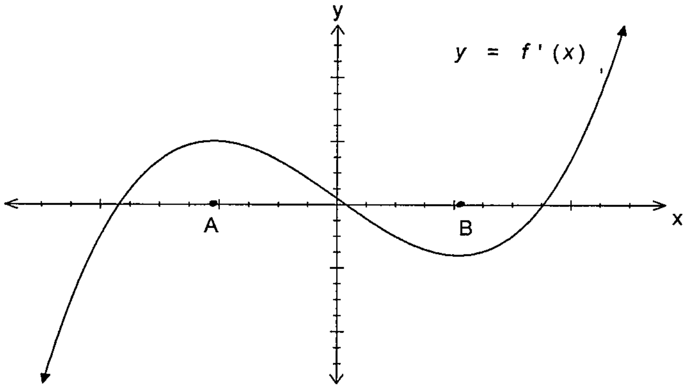
x		1·0	1·5	2·0	2·5	3·0
$f(x)$		1.3	1.8	2.1	1.2	0.6

By applying the trapezoidal rule with 5 function values, estimate $\int_1^3 f(x)\,dx$

Question 9 (8 Marks) **Marks**

a) Find the equation of the circle with centre (4,-3) and radius 9 units. **1**

b) The sketch below shows the graph of $y = f'(x)$.



i) Copy the sketch of $y = f'(x)$ on to your answer sheet.

ii) Explain why points of inflection exist on $y = f(x)$ at x = A and x = B. **1**

iii) On the same set of axes as $y = f'(x)$, draw a curve $y = f(x)$ that has the features indicated by $y = f'(x)$. **2**

c) A woman invests \$5000 into a superannuation package at the beginning of each year. The money invested earns 8% p.a. interest. What is the value of the package after 20 years? **3**

Question 10 (7 Marks)

a) Evaluate $\sum_{k=2}^{5} 2^k - 1$ **1**

b) A local car dealer was offering a loan on the purchase of a \$20 000 car over 5 years. As an incentive to buy, they were advertising "no interest" on the first 6 months. After the first 6 months, interest of $1\frac{1}{2}$% per month on the amount owing was charged. A customer agrees to make 60 monthly repayments of \$M, with the first instalment due 1 month after the purchase of the car.

If A_n is the amount owing after n months, then

i) Find an expression for A_6. **1**

ii) Show that $A_8 = (20\,000 - 6M) \times 1{\cdot}015^2 - M(1 + 1{\cdot}015)$. **2**

iii) Hence find an expression for A_{60}. **1**

iv) Find the value of M. **2**

END OF EXAM

CHS Mathematics Semester 1 2009

Question 1

a) i) $y = \frac{x^4}{2} - 3x + c$

ii) $f(x) = \frac{-4}{x} + c$

iii) $y = \frac{3x^{4/3}}{4} + c$

b) $S_n = \frac{n}{2}(a + l)$

$= \frac{31}{2}(6 + 156)$

$= \underline{\underline{2511}}$

c) $a + (a+d) = 20$, $d = 4$

$2a = 16$

$a = 8$

$8 + 12 + 16 = \underline{\underline{36}}$

Question 2

a) $y = 4x^3 - 2x^{\frac{1}{2}}$

$y' = 12x^2 - x^{-\frac{1}{2}}$

$y'' = \underline{\underline{24x + \frac{1}{2}x^{-\frac{3}{2}}}}$ or $24x + \frac{1}{2\sqrt{x^3}}$

b) $y = 5x - x^2$

$y' = 5 - 2x$, at $x = 2$

$m = 5 - 4$

$\underline{\underline{m = 1}}$

c) $x^2 = 2y$ $\therefore a = \frac{1}{2}$

Focus $\underline{\underline{S(0, \frac{1}{2})}}$

d) $\int_1^3 (3x^4 + 1)\,dx = \left[\frac{3x^5}{5} + x\right]_1^3$

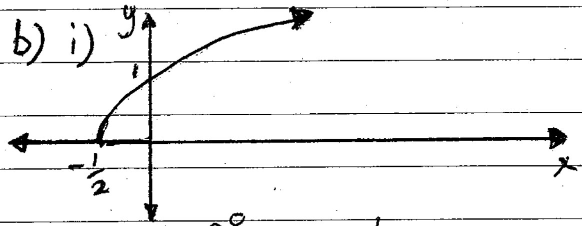
$= \left[\left(\frac{3 \times 3^5}{5} + 3\right) - \left(\frac{3 \times 1^5}{5} + 1\right)\right]$

$= \underline{\underline{147.2}}$

Question 3

a) $\int 5 + x^{-2}\,dx = \underline{\underline{5x - \frac{1}{x} + c}}$

b) i)



ii) $A = \int_{-\frac{1}{2}}^{0} (2x+1)^{\frac{1}{2}}\,dx$

$= \left[\frac{(2x+1)^{\frac{3}{2}}}{\frac{3}{2} \times 2}\right]_{-\frac{1}{2}}^{0}$

$= \left[\frac{1}{3} - 0\right]$

$= \frac{1}{3}$ //

Question 4

a) $\int (2x+5)^3\,dx = \frac{(2x+5)^4}{8} + c$

b) $y = x^3 - 3x^2 + 2$

$y' = 3x^2 - 6x$

$y'' = 6x - 6$

i) Stat pts when $y' = 0$

$3x^2 - 6x = 0$

$3x(x-2) = 0$ $\therefore \underline{x = 0}$, $\underline{x = 2}$

ii) $x = 0$ $y'' = -6$ $\therefore$ max TP at $(0, 2)$

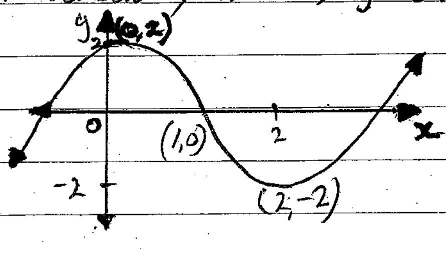
$x = 2$ $y'' = 6$ $\therefore$ min T.P. at $(2, -2)$

iii) Inflection when $y'' = 0$

$6x - 6 = 0$ $\therefore x = 1$

from part ii) we can see a change in concavity at $x = 1$, $y = 0$.

iv)



Question 5

a) $A = \$60000 \times (0.85)^{10}$

$= \$11\,812.46$

b) i) $T_n = ar^{n-1}$

$= 90 \times 0.8^4$

$= 36.864\text{ cm}$

ii) $h = 120 + S_5$

$= 120 + \dfrac{90(1-0.8^5)}{1-0.8}$

$= 422.544\text{ m}$

iii) Max $h = 120 + S_\infty$

$= 120 + \dfrac{90}{1-0.8}$

$= 570\text{ cm}$

$= 5.7\text{ m}$

Question 6

a) $x^2 = 8(y-1)$ $\quad a = 2$

i) $V(0,1)$

ii) $S(0,3)$

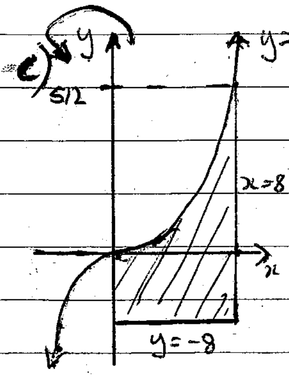
iii) $y = -1$

b) $(x-2)^2 + (y-3)^2 = (y+1)^2$

$(x-2)^2 + y^2 - 6y + 9 = y^2 + 2y + 1$

$(x-2)^2 = 8y - 8$

c)



$V = \pi r^2 h - \pi \int_0^{512} x^2\,dy$

$= \pi \times 8^2 \times 520 - \pi \int_0^{512} y^{\frac{2}{3}}\,dy$

$= 33280\pi - \pi\left[\frac{3}{5} y^{\frac{5}{3}}\right]_0^{512}$

$= 33280\pi - \frac{3\pi}{5}\left[512^{\frac{5}{3}} - 0^{\frac{5}{3}}\right]$

$= 13619\frac{1}{5}\pi\ u^3$

$\doteqdot 42786\ u^3$

Question 7

a) $(x+1)^2 + (y-2)^2 = (x-5)^2 + (y-4)^2$

$x^2 + 2x + 1 + y^2 - 4y + 4 = x^2 - 10x + 25 + y^2 - 8y + 16$

$12x + 4y = 36$

$3x + y = 9$

b) i) $x + y - 1 = 0$

$y = 3 - x^2$

$x + (3 - x^2) - 1 = 0$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0 \quad \therefore x = 2,\ x = -1$

$y = -1,\ y = 2$

ii) $A = \int_{-1}^{2} \left[(3-x^2) - (1-x)\right] dx$

$= \int_{-1}^{2} (-x^2 + x + 2)\,dx$

$= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^{2}$

$= \left[\left(-\frac{8}{3} + \frac{4}{2} + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)\right]$

$= \left(3\frac{1}{3}\right) - \left(-1\frac{1}{6}\right)$

$= 4\frac{1}{2}\ u^2$

Question 8

a) $f(x) = x^3 - 12x$

$f'(x) = 3x^2 - 12 \qquad f'(x) > 0$

$3x^2 > 12$

$x^2 > 4 \qquad x < -2 \text{ or } x > 2$

b) $x + y = 24$

$f(x) = x^2 + (24-x)^2$

$= 2x^2 - 48x + 576$

$f'(x) = 4x - 48 \qquad P'(x) = 0,\ x = 12$

$f''(x) = 4 \therefore$ min at $x = 12$, $y = 12$

c) $I \doteqdot \frac{h}{2}[\text{first} + \text{last} + 2 \times \text{others}]$

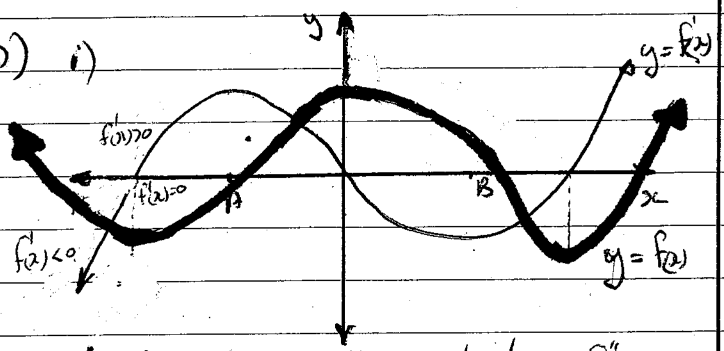
$= \frac{0.5}{2}\left[1.3 + 0.6 + 2(1.8 + 2.1 + 1.2)\right]$

$= 3.025$

Question 9

a) $(x-4)^2 + (y+3)^2 = 81$

b) i)



ii) At A and B $y'' = 0$, but $y'' = f''(x)$ so $f''(x) = 0$ and y' changes direction either side of A, B ∴ concavity changes on $y = f(x)$ at A and B.

c) $A_1 = 5000 \times 1.08^{20}$

$A_2 = 5000 \times 1.08^{19}$

$\vdots$

$A_n = 5000 \times 1.08^{n}$

$S_n = 5000[1.08 + 1.08^2 + \ldots + 1.08^{20}]$

$= 5000\left[\frac{1.08(1.08^{20}-1)}{1.08-1}\right]$

$= \$247\ 114.61$

Question 10

a) $(2^2-1) + (2^3-1) + (2^4-1) + (2^5-1)$

$= 3 + 7 + 15 + 31$

$= 56$

b) i) $A_1 = \$20\ 000 - M$

$A_2 = \$20\ 000 - 2M$

$\vdots$

$A_6 = \$20\ 000 - 6M$

ii) $A_7 = A_6 \times 1.015 - M$

$= (\$20000 - 6M) \times 1.015 - M$

$A_8 = A_7 \times 1.015 - M$

$= (\$20000 - 6M) \times 1.015^2 - 1.015M - M$

$= (20\ 000 - 6M) \times 1.015^2 - M(1 + 1.015)$

iii) $A_{60} = (20\ 000 - 6M) \times 1.015^{54} - M(1 + 1.015 + \ldots + 1.015^{53})$

iv) $A_{60} = 0$

$20\ 000 \times 1.015^{54} - 6 \times 1.015^{54} M - M(1 + 1.015 + \ldots + 1.015^{53}) = 0$

$C - (B \times M) - (M \times A) = 0$

Now $1 + 1.015 + 1.015^2 + \ldots + 1.015^{53} = S_n$ of G.P.

$= \frac{1(1.015^{54} - 1)}{1.015 - 1}$

$= 82.295 \rightarrow A$ (memory)

$6 \times 1.015^{54} = 13.41 \rightarrow B$

$20\ 000 \times 1.015^{54} = 44\ 688.55 \rightarrow C$

So

$C - B \times M - A \times M = 0$

$C = BM + AM$

$= M(A+B)$

$M = \frac{C}{A+B}$

$\doteqdot 467$